Resonant grazing bifurcations in simple impacting systems

Indranil Ghosh and David J.W. Simpson

School of Mathematical and Computational Sciences
Massey University, New Zealand

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Impact Oscillators

- Many engineering systems involve vibrations and impacts.

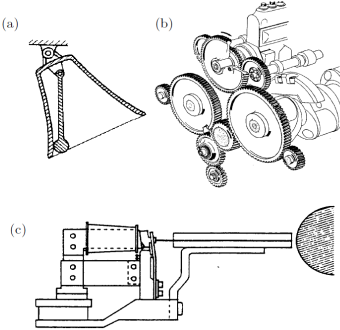

Figure: Examples of simple impacting systems: (a) a bell, (b) a gear assembly, (c) an impact print hammer. Picture taken from di Bernardo *et al.* (2008)

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- Mechanical devices are often engineered with loose-fitting joints to accommodate thermal expansion, and the dynamics of this often lead to impacts in the joint.

Literature Survey (the 80's & 90's)

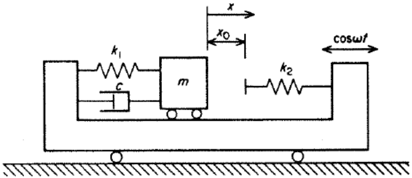


Figure 1. The physical system.

(a) S.W. Shaw *et al.*, 1983.

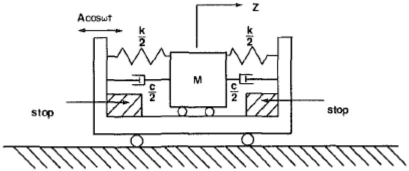
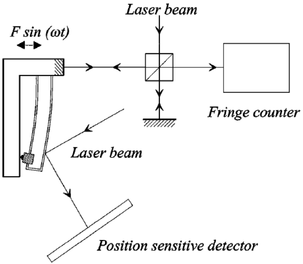


Fig. 1 The physical model

(b) S.W. Shaw , 1985.



(c) J. de Weger *et al.*, 1996.

Literature Survey (the 2000's)

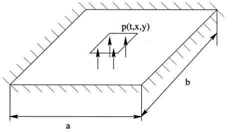


Fig. 1. The geometry of a rectangular plate under transversal forcing.

(a) J. Qiu and Z.C. Feng, 2000.

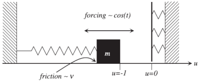


Figure 2. The physical system studied in this paper. The mass m of the oscillator can collide with a yielding wall. We assume that the massless wall is attached with frictionless springs to the fixed world. The resilience of the wall is determined by the stiffness of the second spring. We use normalized units with $m = 1$, the wall position at rest at $u = 0$, and the rest position of the oscillating mass at $u = -1$.

(b) J. Molenaar *et al.*, 2001.

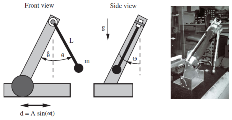


Fig. 1. Schematic (left and middle) and photo (right) of the pendulum/impact barrier assembly.

(c) P. T. Piiroinen *et al.*, 2004.

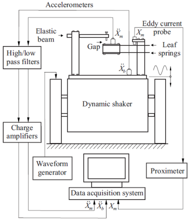
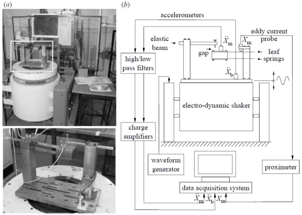
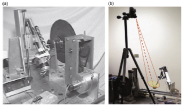


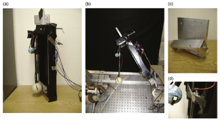
FIG. 1. Schematic diagram of the experimental rig [3].

(d) S. Banerjee *et al.*, 2009.



(e) J. Ing *et al.*, 2011.





(f) T. Witelski *et al.*, 2014.

An experimental example

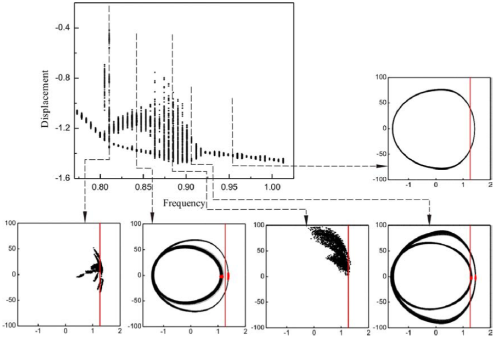


Figure: Bifurcation diagram obtained from the paper by Pavlovskaia *et al.*, 2010.

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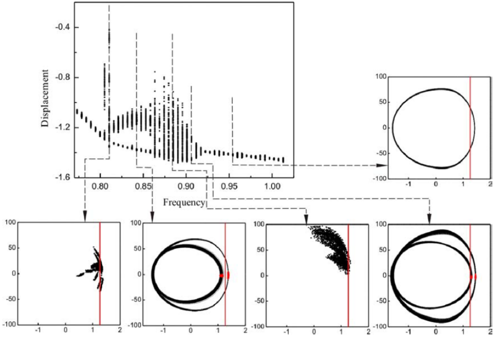

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- Why does a stable period-two solution appear so close to grazing?

A linear oscillator with hard impacts

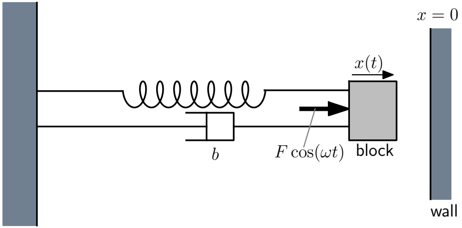

Figure: Equations: $\ddot{x} + b\dot{x} + x + 1 = F\cos(\omega t)$ and $\dot{x} \mapsto -r\dot{x}$ whenever $x = 0$. The oscillator is under-damped $(0 < b < 2)$. Let F be the primary bifurcation parameter and ω be the second.

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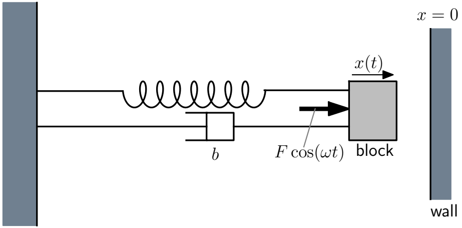

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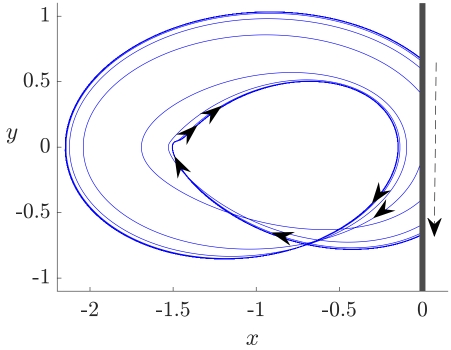
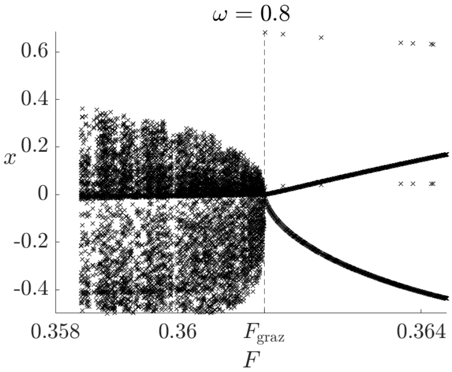
- If the block hits the wall with zero velocity, this is a *grazing* impact.
- A grazing bifurcation occurs when the limit cycle has a grazing impact.

Typical phase portrait and bifurcation diagram





Grazing occurs at $F = F_{\text{graz}}(\omega)$, where $F_{\text{graz}}(\omega) = \sqrt{(1-\omega^2)^2 + b^2\omega^2}$.

Two-parameter bifurcation diagram

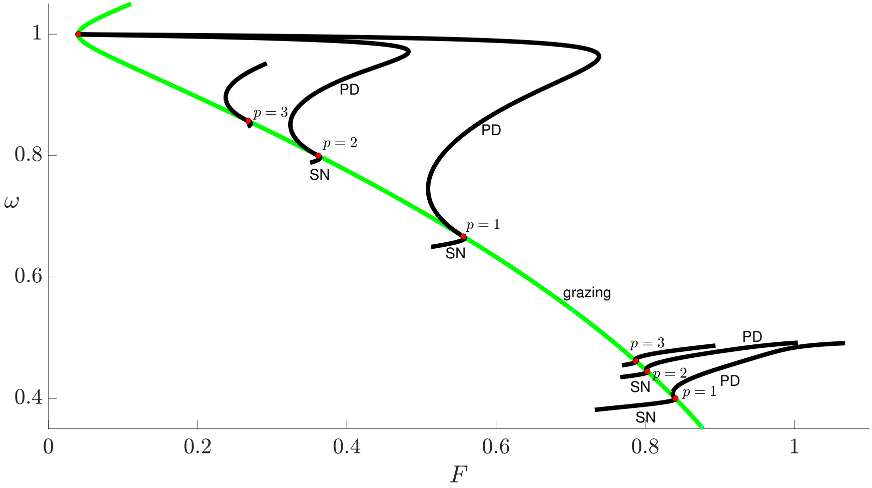
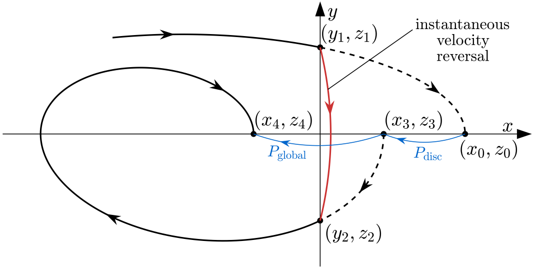


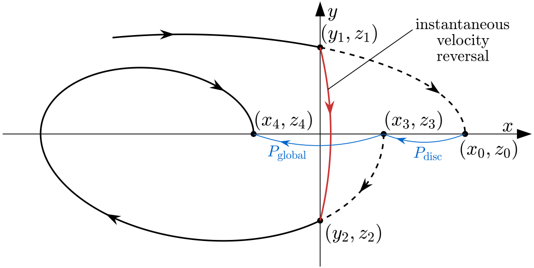
Figure: See Ivanov (1993), and Nordmark (2001).

Poincaré map



- Let $y(t) = \dot{x}(t)$ and $z = (t - t_{\text{ref}}) \mod \frac{2\pi}{\omega}$.

Poincaré map



- Let $y(t) = \dot{x}(t)$ and $z = (t - t_{\text{ref}}) \mod \frac{2\pi}{\omega}$.
- Use $y = 0$ as the Poincaré section. The map: $(x', z') = P(x, z)$ where $P = P_{\text{global}} \circ P_{\text{disc}}$.

Poincaré map

- For a parameter $\mu \in \mathbb{R}$,

$$P_{\text{global}}(x, z; \mu) = A \begin{bmatrix} x \\ z \end{bmatrix} + q\mu + \mathcal{O}((|x|+|z|+|\mu|)^2),$$

  where $A = \mathrm{D}P_{\text{global}}(0, 0; 0)$, and $q = \frac{\partial P_{\text{global}}}{\partial \mu}(0, 0; 0)$.
- The discontinuity map by Nordmark is given by

$$P_{\text{disc}}(x, z; \mu) = \begin{cases} \begin{bmatrix} x \\ z \end{bmatrix}, & x \leq 0, \\ \begin{bmatrix} r^2 x + \tilde{O}(3) \\ z - \frac{\sqrt{2}}{\omega}(1 + r)\sqrt{x} + \tilde{O}(2) \end{bmatrix}, & x > 0. \end{cases}$$

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- For a period-p solution of our map P with one impact, the MPS is the fixed point of $P_{\text{global}}^{p} \circ P_{\text{disc},R}$.
- Since the MPS is smooth, standard numerical methods like Newton’s method can be used to follow fixed points while $x > 0$.
- We continue zeros of the function $G = P_{\text{global}}^{p} \circ P_{\text{disc},R} - I$.

Numerics

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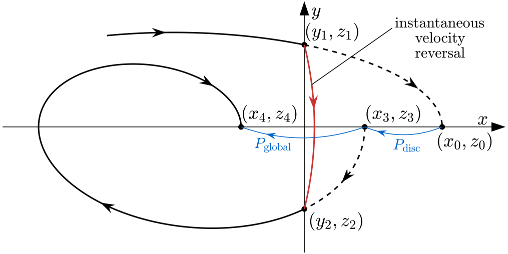

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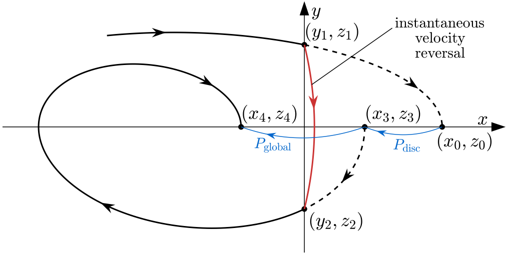

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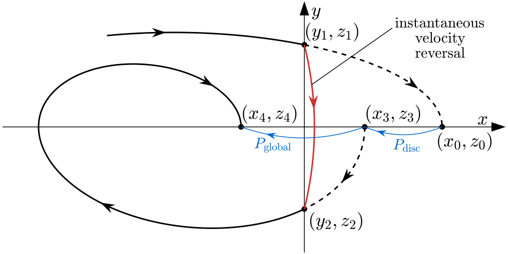

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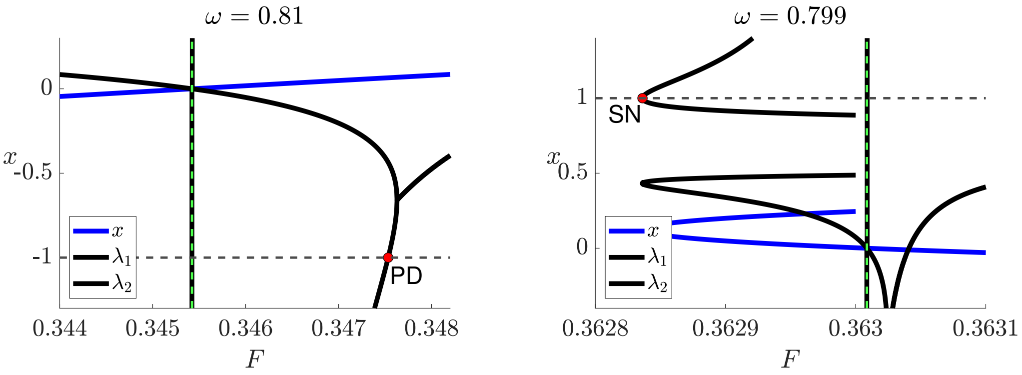
- This function is smooth in a neighborhood of $(y_1, z_1) = (0, 0)$.

One-parameter bifurcation diagrams



Resonance

- Branches of MPSs emanate from the grazing bifurcation, either to the left or the right, and Nordmark (*Nonlinearity*, 2001) showed that this is determined by the values of τ and δ.

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- Here

$$\tau = 2e^{-\frac{\pi b}{\omega}} \cos\left(\frac{2\pi\xi}{\omega}\right), \qquad \delta = e^{-\frac{2\pi b}{\omega}}.$$

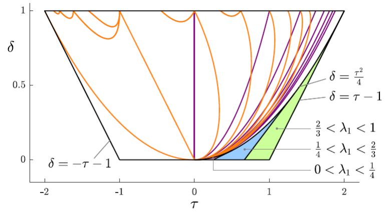

Figure: Division of the (τ, δ) plane.

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Lemma

For the linear impact oscillator,

i) *for* $p = 1$*, we have* $a_{1,2} = 0$ *if and only if* $\frac{\xi}{\omega^*} = \frac{n}{2}$*, for some* $n \in \mathbb{Z}$*;*

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ii) *for* $p \geq 2$, *we have* $\tau = 2\sqrt{\delta}\cos\left(\frac{\pi}{p}\right)$ *if and only if* $\frac{\xi}{\omega^*} = n \pm \frac{1}{2p}$, *for some* $n \in \mathbb{Z}$

Asymptotics ($p = 1$)

Let $\eta = \omega - \omega^*$. For $p = 1$ let

$$c_{\pm,1} = \mp\left(1 + \phi^2\delta^p\right) + a_{11}\phi^2 + a_{22} + \frac{\alpha^2\ell}{(1 - a_{22})\gamma}.$$

Then

$$g_{\pm,1}(\eta) = \frac{\alpha^2\left(\frac{da_{12}}{d\eta}\right)^2}{\beta\gamma c_{\pm,1}}\eta^2 + \mathcal{O}\left(\eta^3\right).$$

Asymptotics $(p \geq 2)$

Let

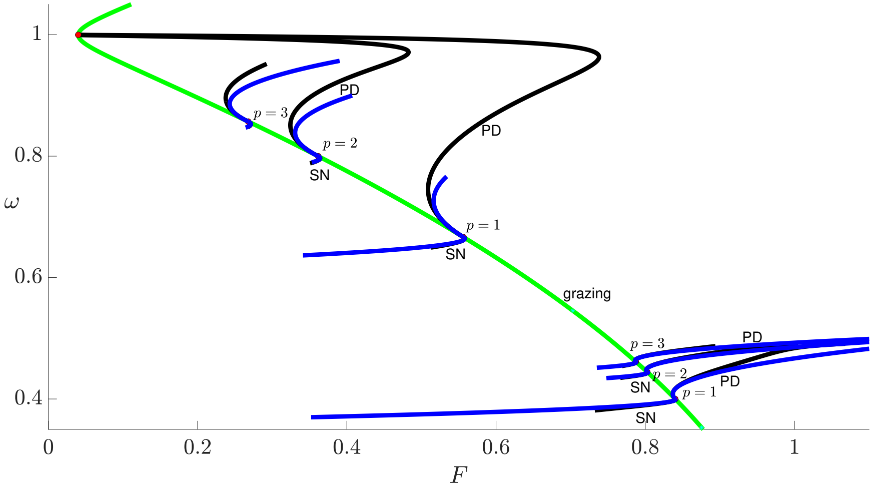
$$c_{\pm,p} = \mp\left(1+\phi^2\delta^p\right) - \sqrt{\delta^p}\left(1+\phi^2\right) + \frac{\alpha^2\ell}{\left(1+\sqrt{\delta^p}\right)\gamma}.$$

Then

$$g_{+,p}(\eta) = \frac{\alpha^2 a_{12}^2 p^2 \delta^{p-2}(\delta-\tau+1)\left(\kappa'\right)^2}{8\sin^4\left(\frac{\pi}{p}\right)\left(1+\sqrt{\delta^p}\right)^2\beta\gamma c_{+,p}}\,\eta^2 + \mathcal{O}\left(\eta^3\right),$$

$$g_{-,p}(\eta) = \frac{\alpha^2 a_{12}^2 p^2 \delta^{p-2}(\delta-\tau+1)\left(\kappa'\right)^2\left(1-\frac{c_{+,p}}{2c_{-,p}}\right)}{4\sin^4\left(\frac{\pi}{p}\right)\left(1+\sqrt{\delta^p}\right)^2\beta\gamma c_{-,p}}\,\eta^2 + \mathcal{O}\left(\eta^3\right).$$

Two-parameter bifurcation diagram with asymptotics



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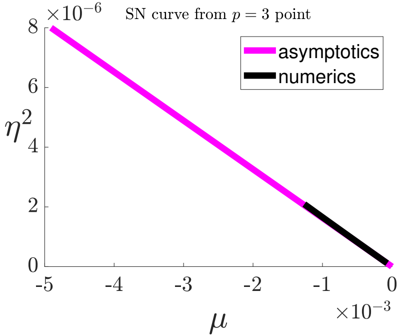

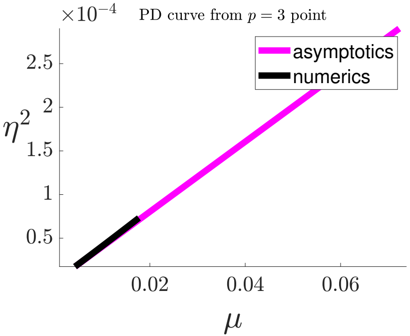

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- We have also theoretically come up with matching asymptotics, unfolding the codimension-two points.
- Future: More complete bifurcation diagram, other bifurcation curves.

The End

Thank you! Questions?